ON EIGENVALUES OF SYMMETRIC $(+1, -1)$ MATRICES[†]

BY

A. J. HOFFMAN

ABSTRACT

To every symmetric matrix A with entries ± 1, we associate a graph $G(A)$, and ask (for two different definitions of distance) for the distance of $G(A)$ to the nearest complete bipartite graph (cbg). Let $\lambda_1(A)$, $\lambda^1(A)$ be respectively the algebraically largest and least eigenvalues of A. The Frobenius distance (see Section 4) to the nearest cbg is bounded above and below by functions of $n - \lambda_1(A)$, where $n =$ ord A. The ordinary distance (see Section 1) to the nearest cbg is shown to be bounded above and below by functions of $\lambda^1(A)$. A curious corollary is: there exists a function f (independent of n, and given by (1.1)), such that $|\lambda_i(A)| \leqq f(\lambda^1(A)$, where $\lambda_i(A)$ is any eigenvalue of A other than $\lambda_1(A)$.

1. Introduction

Let A be a real symmetric matrix; denote its eigenvalues in descending order by $\lambda_1(A) \geqq \lambda_2(A) \geqq \cdots$, in ascending order by $\lambda^1(A) \leqq \lambda^2(A) \leqq \cdots$. We shall be concerned with the case where every entry in A is ± 1. To simplify exposition and some calculations, we shall assume all diagonal entries are $+1$, though this assumption is not essential. We shall show the following curiosity: $\lambda^1(A)$ gives bounds on all other eigenvalues which are (except for $\lambda_1(A)$, of course) independent of the order of A. Specifically, define

$$f(x) = 4(2^{x^2-x+1}(x^2 - x) - x + 3). \tag{1.1}$$

THEOREM 1.1. *Let $A = (a_{ij})$ be a symmetric matrix with $a_{ii} = 1$ for all i and $a_{ij} = \pm 1$ for all i,j; then*

$$|\lambda_i(A)| \leqq f(\lambda^1(A)) \text{ for } i > 1. \tag{1.2}$$

[†] This work was supported (in part) by the U. S. Army under contract # DAHC04–C–0023.
Received December 15, 1973

If the order of A is n, then

$$0 \leqq n - \lambda_1(A) \leqq f(\lambda^1(A)).$$

Theorem 1.1 follows readily from Theorem 1.2, which relates $\lambda^1(A)$ to a property of a graph associated with A. A graph G, with vertex set $V = V(G)$ is said to be a complete bipartite graph if V can be partitioned into disjoint subsets V_1 and V_2 (one of which may be empty) so that every pair of vertices in the same V_i are not adjacent, every pair of vertices in different V_i are adjacent. We also need the concept of distance between graphs. If G and H are two graphs on the same set of vertices then $d(G,H) = d$ if d is the smallest number such that for each vertex i, the number of edges in G adjacent to i which are not edges of H, plus the number of edges in H adjacent to i which are not edges of G, does not exceed d. Alternatively, $d(G,H)$ is the maximum valence of the graph of edges added and deleted to convert G to H.

THEOREM 1.2. *If A satisfies the hypothesis of Theorem 1.1, if A is of order n, and if $G(A)$ is the graph with $V(G(A)) = \{1,\cdots,n\}$, and i and j adjacent if and only if $a_{ij} = -1$, then there is a complete bipartite graph on n vertices whose distance to G is at most $\frac{1}{2}f(\lambda^1(A))$.*

In Section 4, we shall present analogous results emphasizing $n - \lambda_1$ rather than λ^1.

2. Proof of Theorem 1.2

LEMMA 2.1. *Let $\lambda = \lambda^1(A)$, and assume*

$$n > \frac{\lambda(\lambda-1)}{2}. \tag{2.1}$$

Then neither B_1 nor B_2 can be a principal submatrix of A, where

$$B_1 = \begin{pmatrix} 1 & 1 \cdots 1 & 1 \cdots 1 \\ 1 & & \\ \vdots & A_1 & -J \\ 1 & & \\ 1 & & \\ \vdots & -J & A_2 \\ 1 & & \end{pmatrix} \qquad B_2 = \begin{pmatrix} 1 & 1 \cdots 1 & -1 \cdots -1 \\ 1 & & \\ \vdots & A_1 & J \\ 1 & & \\ -1 & & \\ \vdots & J & A_2 \\ -1 & & \end{pmatrix}$$

where J is a square matrix of order n, every entry of which is 1.

Proof. Consider first B_1. Let y be the vector of order $2n+1$ whose first coordinate is 1, all other coordinates are the real number x. Examine the Rayleigh quotient $y'B_1y/y'y$. Since the entries in A_1 and A_2 are ± 1, it follows that $y'B_1y \leqq 1 + 4nx$. Hence,

$$\frac{y'B_1y}{y'y} \leqq \frac{1+4nx}{1+2nx^2}. \tag{2.2}$$

Suppose we can show that for some $\varepsilon > 0$, there exists a real number x such that

$$\frac{1+4nx}{1+2nx^2} = \lambda - \varepsilon. \tag{2.3}$$

Then (2.2) and (2.3), combined with $\lambda^1(B_1) \leqq y'B_1y/y'y$ will imply $\lambda^1(B_1) \leqq \lambda - \varepsilon < \lambda^1(A)$. But if B_1 is a principal submatrix of A, $\lambda^1(B_1) \geqq \lambda^1(A)$. So all we need to do is prove (2.3). But solving (2.3) for x shows that a real x exists provided

$$n \geqq \frac{(\lambda-\varepsilon)(\lambda-\varepsilon-1)}{2}, \tag{2.4}$$

which follows from (2.1).

The matrix B_2 is similar to B_1. Hence (2.1) implies $\lambda^1(B_2) < \lambda^1(A)$, a contradiction.

Lemma 2.2. *Let A be any ± 1 matrix with $2m-1$ rows and $2^{2m-1}(m-1)+1$ columns. Then A contains a square submatrix of order m all entries of which are the same.*

Proof. We shall do the case $m = 3$, which is sufficient to indicate the proof in general. The number of columns of A is 65, so at least 33 entries in the first row are the same. Examing the corresponding 33 entries in the second row, at least 17 are the same. Of the corresponding entries in the third row at least 9 are the same; of the corresponding entries in the fourth row at least 5 are the same; of the corresponding entries in the fifth row at least 3 are the same sign. So we have a 5×3 submatrix of A in which each row consists of the same entries. At least 3 of the rows must be the same.

Next, let 1 and 2 be vertices of $G(A)$. Let V_0 be the set of all other vertices of $G(A)$ each of which is adjacent to either none or both of $\{1,2\}$. Let V_1 be the set of all other vertices of $G(A)$ each of which is adjacent to exactly one of $\{1,2\}$. Thus $V(G(A)) = \{1,2\} \cup V_0 \cup V_1$.

LEMMA 2.3. *With the above notation, it is impossible that both*

$$|V_i| \geq 2^{\lambda^2-\lambda}(\lambda^2-\lambda)+1, \; i = 0, 1, \text{ where } \lambda = \lambda^1(A). \tag{2.4}$$

PROOF. Assume otherwise. By a diagonal similarity transformation (which does not change the eigenvalues of A), we may assume that all vertices in V_0 are adjacent to neither 1 nor 2, and all vertices in V_1 are adjacent to 1. Then a principal submatrix of A has the following appearance:

$$A = \begin{bmatrix} 1 & \pm 1 & 1 \cdots 1 & -1 \cdots -1 \\ \pm 1 & 1 & 1 \cdots 1 & 1 \cdots \;\; 1 \\ 1 & 1 & & \\ \vdots & \vdots & A_1 & A_2 \\ 1 & 1 & & \\ -1 & 1 & & \\ \vdots & \vdots & A_2^T & A_3 \\ -1 & 1 & & \end{bmatrix} \tag{2.5}$$

where each A_i is square, of order $2^{\lambda^2-\lambda}(\lambda^2-\lambda)+1$. By Lemma 2.3, applied to A_2, A has a principal submatrix B of the form

$$B = \begin{bmatrix} 1 & \pm 1 & 1 \; \cdots \; 1 & -1 \cdots -1 \\ \pm 1 & 1 & 1 \; \cdots \; 1 & 1 \cdots \;\; 1 \\ 1 & 1 & & \\ \vdots & & A_4 & \pm J \\ 1 & 1 & & \\ -1 & 1 & & \\ \vdots & \vdots & \pm J & A_5 \\ -1 & 1 & & \end{bmatrix}$$

where J is a square matrix of order $(\lambda^2-\lambda)/2+1$, every entry of which is 1. If the off diagonal is $+J$, then deleting the second row and column from B produces a matrix of the form B_2 in Lemma 2.1. If it is $-J$, deleting the first row and column produces a matrix of the form B_1 considered in Lemma 2.1. In both cases, Lemma 2.1 provides the desired contradiction.

Let $s = 2^{\lambda^2-\lambda}(\lambda^2 - \lambda) + 1$. For the remainder of the proof, assume the order of A is at least $3s + 2$ (otherwise, Theorem 1.2 is obviously true). Say that two vertices of $G(A)$ are 0-related if the corresponding V_0 has cardinality at least $2s + 1$, and they are 1-related if the corresponding V_1 has cardinality at least $2s + 1$. By Lemma 2.3, any two vertices are 0-related or 1-related, but not both.

LEMMA 2.4. *Let A be of order n, and let $\tilde{G}(A)$ be the graph whose vertices are the vertices of $G(A)$, with two vertices adjacent if and only if they are 1-related. Then $\tilde{G}(A)$ is a complete bipartite graph.*

PROOF. We must show that:

(2.6) if 1 and 2 are 0-related, 2 and 3 are 0-related, then 1 and 3 are 0-related; and

(2.7) if 1 and 2 are 1-related, 2 and 3 are 1-related, then 1 and 3 are 0-related.

We shall prove (2.6). The proof of (2.7) is analogous. Let W be the set of vertices other than $\{1,2,3\}$, and for each of the eight choices $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, where $\varepsilon_i = \pm 1$, let $W(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ be the cardinality of the set of vertices j in W such that $a_{ij} = \varepsilon_i$, $i = 1,2,3$. We must show that

(2.8) $1 + W(1,1,-1) + W(1,-1,-1) + W(-1,1,1) + W(-1,-1,1) < 2s + 1.$

We have from the fact that 1 and 2 are 0-related and from Lemma 2.3 that

(2.9) $\quad W(1,-1,1) + W(1,-1,-1) + W(-1,1,1) + W(-1,1,-1) < s.$

From the fact that 2 and 3 are 0-related, we have

(2.10) $\quad W(1,1,-1) + W(-1,1,-1) + W(1,-1,1) + W(-1,-1,1) < s.$

Adding (2.9) and (2.10), we obtain (2.8).

Let the two parts into which the vertices of $\tilde{G}(A)$ are partitioned, in accordance with Lemma 2.4, be denoted by W_1 and W_2.

LEMMA 2.5. *If $k \in W_i (i = 1,2)$, then the number of vertices of W_i to which it is adjacent (in $G(A)$) is at most $1 + 2s - \lambda$; the number of vertices of W_j $(j \neq i)$ to which it is not adjacent is at most $1 + 2s - \lambda$.*

PROOF. We shall only prove the first clause. The proof of the second is analogous. Assume $1 \in W_1$ is adjacent to $t > 1 + 2s - \lambda$ vertices in W_1. Then consider the principal submatrix B of order $t + 1$ of A formed by 1 and these vertices.

Since these vertices are 0-related to 1, the row of B corresponding to each of these vertices contains at most s 1's. The first row of B contains exactly one 1. Let $u = (1,\cdots,1)$. Then $(u'Bu)/u'u \leqq (-t^2 + 2 + 2ts)/t + 1 \leqq 1 - t + 2s$. If $t > 1 + 2s - \lambda$, this would imply $(u'Bu)/(u'u) < \lambda$, which is impossible.

Clearly, Lemma 2.5 completes the proof of Theorem 1.2.

3. Proof of Theorem 1.1

By Theorem 1.2, we can write $A = P + B$, where B has the form

$$\left(\begin{array}{c|c} + & - \\ \hline - & + \end{array}\right).$$

Hence, $\lambda_1(B) = n$, $\lambda_i(B) = 0$ for $i > 1$. P is a symmetric matrix in which the sum of the absolute values of the entries in each row is at most $4(1 + 2s - \lambda)$. Hence, $\lambda_1(P) \leqq 4(1 + 2s - \lambda)$, and $-\lambda^1(P) \leqq 4(1 + 2s - \lambda)$. Since $\lambda_2(B) = 0$, and $\lambda_2(A) \leqq \lambda_2(B) + \lambda_1(P)$, we have (1.2), for if $i > 1$, $|\lambda_i(A)| \leqq \max\{\lambda_2(A), -\lambda^1(A)\}$. Since $\lambda_1(A) \geqq \lambda^1(P) + \lambda_1(B)$, and $\lambda_1(B) = n$, we have (1.3). (We have used the Courant-Weyl inequalities on eigenvalues of the sum of symmetric matrices.)

The function f given by (1.1) may very well be too large. We know that any f for which the theorems are valid must be at least $O(x^2)$, but that leaves a considerable gap.

4. Analogous results for $n - \lambda_1$

In Section 1, we defined the distance between two graphs G and H as the maximum valence of the graph $(G - H) \cup (H - G)$. Let us consider a different distance $d_F(G,H)$ (F for Frobenius, for reasons which will be apparent below); $d_F(G,H)$ is the average valence of vertices of the graph $(G - H) \cup (H - G)$. For any matrix A of order n satisfying the hypothesis of Theorem 1.1, let $D_F(A)$ be the smallest Frobenius distance to a complete bipartite graph.

THEOREM 4.1. *If a satisfies the hypothesis of Theorem* 1.1, *then*

$$\frac{D_F(A)}{2} \leqq n - \lambda_1(A) \leqq 2D_F(A). \tag{4.1}$$

To prove the right-hand inequality of (4.1), let $S \cup T$ be a partition of $\{1,\cdots,n\}$ so that the Frobenius distance to $G(H)$ of the corresponding complete bipartite graph is $D_F(A)$. Let $u = (u_1, u_2, \cdots u_n)$, where $u_i = 1$ if $i \in S$, $u_1 = -1$ if $i \in T$.

Then

$$\lambda_1(A) \geqq \frac{1}{n}(n^2 - 2nD_F(A)) = \frac{u'Au}{u'u}. \tag{4.2}$$

To prove the left-hand inequality, let $\tilde{A}$ be the result of transforming A by a diagonal similarity matrix, in which each diagonal entry is ± 1, so that $\tilde{A}x = \lambda_1(A)x$, $x \geqq 0$, $x \neq 0$. Let nE be the number of -1's in $\tilde{A}$, so that

$$E \geqq D_F(A). \tag{4.3}$$

Let B be the matrix obtained from $\tilde{A}$ by replacing every -1 by 0. Then $Bx \geqq \lambda_1(A)x$, so that, according to the Perron-Frobenius theory of non-negative matrices,

$$\lambda_1(B) \geqq \lambda_1(A). \tag{4.4}$$

But $(\lambda_1(B))^2 \leqq \Sigma_i(\lambda_i(B))^2 = \Sigma_{i,j} b_{ij}^2 = n^2 - nE$, so (by (4.3) and (4.4)),

$$(\lambda_1(A))^2 \leqq n^2 - nD_F(A),$$

which yields $n\ D_F W \leqq n^2 - (\lambda_1(A))^2 = (n - \lambda_1(A))\ (n + \lambda_1(A)) \leqq 2n(n - \lambda_1(A))$, which is the left-hand inequality of (4.1).

IBM WATSON RESEARCH CENTER
YORKTOWN HEIGHTS, NEW YORK, U. S. A.